

# HOLONOMY MAP FIBERS OF $\mathbb{CP}^1$ -STRUCTURES IN MODULI SPACE

SHINPEI BABA AND SUBHOJOY GUPTA

ABSTRACT. Let  $S$  be a closed oriented surface of genus  $g \geq 2$ . Fix an arbitrary non-elementary representation  $\rho: \pi_1(S) \rightarrow \mathrm{SL}_2(\mathbb{C})$  and consider all marked (complex) projective structures on  $S$  with holonomy  $\rho$ . We show that their underlying conformal structures are dense in the moduli space of  $S$ .

## 1. INTRODUCTION

Let  $F$  be an oriented connected surface. A (*complex*) *projective structure* on  $F$  is a maximal atlas modeled on  $\hat{\mathbb{C}}$  with transition maps in  $\mathrm{PSL}_2(\mathbb{C})$ . Then each projective structure enjoys a conformal structure induced by  $\hat{\mathbb{C}}$ . Letting  $\tilde{F}$  be the universal cover of  $F$ , a projective structure is equivalently given as a pair of an immersion  $f: \tilde{F} \rightarrow \hat{\mathbb{C}}$  (*developing map*) and a homomorphism  $\rho: \pi_1(F) \rightarrow \mathrm{PSL}_2(\mathbb{C})$  (*holonomy*) such that  $f$  is  $\rho$ -equivariant.

Let  $S$  be a closed oriented surface of genus  $g \geq 2$ , and let  $\mathcal{P}$  be the space of all marked projective structures on  $S$ . Let  $\chi$  be the component of the  $\mathrm{PSL}_2(\mathbb{C})$ -character variety of  $\pi_1(S)$  containing the representations  $\pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{C})$  lifting to  $\pi_1(S) \rightarrow \mathrm{SL}_2(\mathbb{C})$ . The *holonomy map*

$$\mathrm{hol}: \mathcal{P} \rightarrow \chi$$

is the natural forgetful map taking projective structures to their holonomy representations.

It is known that the image of  $\mathrm{hol}$  consists of non-elementary representations in  $\chi$ ; see [GKM00]. (Note that  $\mathrm{Im}(\mathrm{hol})$  contains many nondiscrete representations.) Then a basic question is to understand the *holonomy fiber*  $\mathrm{hol}^{-1}(\rho) =: \mathcal{P}_\rho$  in  $\mathcal{P}$  ([Gol80, Kap95, GKM00, Dum09]). In particular [Hub81, p 274] asked what its projection to the Teichmüller space  $\mathcal{T}$  looks like, given by underlying conformal structures.

In this paper, we consider its further projection into the moduli space  $\mathcal{M}$  of Riemann surfaces:

$$\mathcal{P} \xrightarrow{p} \mathcal{T} \xrightarrow{\pi} \mathcal{M}.$$

We shall prove:

**Theorem 1.1.** *Let  $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{C})$  be any non-elementary homomorphism that lifts to  $\pi_1(S) \rightarrow \mathrm{SL}(2, \mathbb{C})$ . Then the holonomy fiber  $\mathcal{P}_\rho$  projects onto a dense subset of the moduli space  $\mathcal{M}$ .*

In contrast,  $\mathcal{P}_\rho$  projects onto a *discrete* subset of Teichmüller space  $\mathcal{T}$ : Since  $\mathrm{hol}$  is a local homeomorphism ([Hej75]),  $\mathcal{P}_\rho$  is a discrete set of  $\mathcal{P}$ . In addition, for any compact set  $K \subset \mathcal{T}$ , the restriction of  $\mathrm{hol}$  to  $p^{-1}(K)$  is proper (see Theorem 3.2 of [Tan99]). Therefore  $\mathcal{P}_\rho \cap p^{-1}(K)$  is a finite set in  $\mathcal{P}$ , and hence so is its projection  $p(\mathcal{P}_\rho) \cap K$  in  $\mathcal{M}$ .

The proof of Theorem 1.1 utilizes the operation of projective structures, called *grafting*: given a hyperbolic surface  $\tau \in \mathcal{T}$  and a measured geodesic lamination  $L$ , grafting yields a new projective surface  $Gr_L(\tau)$ . If  $L$  is a multiloop, it inserts projective annuli along the components of  $L$ , where the heights of the annulus are given by weights. This gives a geometric parameterization  $\mathcal{P} \cong \mathcal{T} \times \mathcal{ML}$ , called *Thurston coordinates* (see [KT92, KP94]), where  $\mathcal{ML}$  is the set of all measured laminations on  $S$ . In particular, grafting along multiloops weighted by  $2\pi$ -multiples preserves the Fuchsian holonomy (see [Gol87]). Moreover this “ $2\pi$ ”-grafting generalizes to grafting of general projective structures along “admissible” loops (see §2.2), which preserves holonomy as well.

When grafting a hyperbolic surface  $\tau$ , by scaling the transversal measure on the lamination  $L$ , we obtain a ray of projective structures,  $Gr_{tL}(\tau)$  ( $t \geq 0$ ), in  $\mathcal{P}$ . Then it descends to a ray  $\mathrm{gr}_{tL}$  in  $\mathcal{T}$ , and then to a ray in  $\mathcal{M}$ .

Then [Gup] and [Gup14] yield the strong asymptoticity of these grafting rays to Teichmüller geodesic rays, which implies

**Theorem 1.2** (Corollary 1.3 of [Gup]). *For any  $\tau \in \mathcal{T}$  and almost-every  $L \in \mathcal{ML}$ , the grafting ray  $Gr_{tL}(\tau)$  ( $t \geq 0$ ) descends to a dense ray in moduli space.*

Here “almost-every” refers to the Thurston measure on  $\mathcal{ML}$ , the space of measured laminations, so that  $(\tau, L)$  corresponds to Teichmüller rays that are dense in  $\mathcal{M}$  ([Mas82]). The density in Theorem 1.2 yields the density in Theorem 1.1.

In [Gup], the second author obtained the result in Theorem 1.1 for the case when  $\rho$  is Fuchsian— $\rho$  determines a hyperbolic structure  $\tau$  on  $S$ , and a dense subset in  $\mathcal{M}$  is given by  $(2\pi)$ -grafting  $\tau$  along multiloops and creating a sequence that approximates a dense grafting ray given by Theorem 1.2.

Let  $C \cong (\tau, L)$  denote any projective structure on  $S$  in Thurston coordinates. We approximate the grafting ray by  $2\pi$ -grafting:

**Theorem 1.3.** *For every  $\epsilon > 0$ , if a measured lamination  $M$  on  $S$  is sufficiently close to  $L$  in  $\mathcal{ML}$ , then for every sufficiently large  $t > 0$ , there is an admissible*

loop  $N$  on  $C$ , such that,

$$d_{\mathcal{T}}(gr_N(C), gr_{tM}(\tau)) < \epsilon,$$

where  $d_{\mathcal{T}}$  denotes the Teichmüller metric on  $\mathcal{T}$ .

Let  $\rho = \text{hol}(C)$ . Then Theorem 1.3 and Theorem 1.2 imply that  $\mathcal{P}_{\rho}$  projects onto a subset in  $\mathcal{M}$  whose  $\epsilon$ -neighborhood is  $\mathcal{M}$ : Pick a generic lamination  $M$  so that  $(\tau, M)$  correspond to a dense Teichmüller ray in  $\mathcal{M}$ , so that Theorem 1.2 holds. Since  $\epsilon > 0$  is arbitrary, the projection of  $\mathcal{P}_{\rho}$  must be dense in  $\mathcal{M}$ , which proves Theorem 1.1. Note that, in order to create a dense subset in  $\mathcal{M}$ , we only used projective structures obtained by grafting  $C$  along multiloops; in addition we can choose different  $C$  in  $\mathcal{P}_{\rho}$ .

Our strategy is to generalize the argument for Fuchsian holonomy  $\pi_1(S) \rightarrow \text{PSL}(2, \mathbb{R})$  in [Gup], where the second author considers only grafting of hyperbolic structures, and uses the Thurston metric on projective surfaces to construct quasiconformal maps with small distortion. However, in this paper we consider grafting of a general projective structure  $C$  with arbitrary holonomy  $\rho: \pi_1(S) \rightarrow \text{PSL}_2(\mathbb{C})$ , and arguments here lie in three-dimensional hyperbolic geometry. In particular, we introduce smoother piecewise Euclidean/Hyperbolic metrics on projective surfaces, which modify the Thurston metrics; then, in order to prove Theorem 1.3, we construct piecewise smooth biLipschitz maps with the biLipschitz constant close to 1. This new metric is given by fat traintracks on the projective surfaces “supported” on round cylinders on the Riemann sphere. Such fat traintracks were used in [Baba], where the first author relates different projective structures with common holonomy by  $2\pi$ -grafting. Since, however, our projective structures  $\text{Gr}_N(C)$  and  $\text{Gr}_{tM}(\tau)$  in Theorem 1.3 typically do not share holonomy, and we are interested in their conformal structures, different and more geometric treatment is needed.

We note that, as described in [Gol04], the holonomy map  $\text{hol}: \mathcal{P} \rightarrow \chi$  gives a “resolution” of the mapping class group action on  $\chi$ . (The mapping class group action is hol-equivariant and its action on  $\mathcal{P}$  is discrete.) Thus it would be interesting if the holonomy fibers  $\mathcal{P}_{\rho}$  tell us about the action on  $\chi$ .

**Outline of the proof of Theorem 1.3.** If  $M$  is sufficiently close to  $L$ , then there is a *nearly straight* fat traintrack  $T$  on  $\tau$  carrying both laminations. Then we can choose a multiloop  $N$  also carried by  $T$  so that  $L + N$  is a good approximation of  $tM$ , compared on  $T$ .

Accordingly, there are fat traintracks on the projective surfaces  $\text{Gr}_N(C)$  and  $\text{Gr}_{tM}(\tau)$  corresponding to  $T$  on  $\tau$ . Then we construct certain Euclidean/hyperbolic metrics on the projective surfaces: The fat traintracks enjoys Euclidean metrics, given by the carried geodesic measured laminations, and hyperbolic metrics in

their complements, given by  $\tau$ . Then, with respect to those metrics, for sufficiently large  $t$ , we construct a piecewise smooth bilipschitz map from  $\text{Gr}_N(C)$  and  $\text{Gr}_{tM}(\tau)$  with small distortion, which yields a quasiconformal map between  $\text{gr}_N(C)$  and  $\text{gr}_{tM}(\tau)$  with a small distortion.

**Acknowledgements.** The authors would like to thank Yair Minsky. The second author would like to acknowledge support of the Danish National Research Foundation grant DNRF95 (Centre for Quantum Geometry of Moduli Spaces).

## 2. PRELIMINARIES

**2.1. Teichmüller metric on Teichmüller space and moduli space.** We refer to [Ahl66, Hub06] for a background to Teichmüller theory. In this paper, let  $\pi$  be the projection from the Teichmüller space  $\mathcal{T}$  to the moduli space  $\mathcal{M}$  obtained by forgetting markings.

Given two marked Riemann surfaces  $X, Y$ , the *Teichmüller distance* between them is

$$d_{\mathcal{T}}(X, Y) := \frac{1}{2} \log K$$

where  $K$  is the minimum dilatation over all quasiconformal maps between  $X$  and  $Y$  that preserve the marking. This in fact defines a complete Finsler metric on  $\mathcal{T}$  invariant under the mapping class group. Thus it induces a metric topology on moduli space  $\mathcal{M}$  that coincides with any other natural topology obtained from the other standard definitions.

**2.2. Laminations and traintracks.** (See [CEG87, CB88, PH92, Kap01], for background on measured laminations and traintracks.) A *geodesic lamination*  $\lambda$  on a hyperbolic surface is a closed subset that is foliated by simple geodesics (*leaves*). A geodesic lamination is *minimal* if each leaf is dense in the lamination. On the other hand, a geodesic lamination is *maximal* if the complementary regions are ideal triangles. A basic example of a geodesic lamination is a *multiloop* which is a union of disjoint simple closed geodesics. Then, given a geodesic lamination  $\lambda$ , there is a unique decomposition into disjoint components:

$$\lambda = \lambda_1 \sqcup \lambda_2 \sqcup \cdots \sqcup \lambda_k \sqcup l_1 \sqcup l_2 \sqcup \cdots \sqcup l_m \sqcup \gamma$$

where  $\lambda_i$  ( $1 \leq i \leq k$ ) are *minimal* sublaminations,  $l_i$  ( $1 \leq i \leq m$ ) are isolated bi-infinite geodesics that accumulate on the other components, and  $\gamma$  is a multiloop. A *measured lamination* is a geodesic lamination equipped with a measure on transverse arcs that is invariant under homotopies of arcs preserving the transversality.

**Definition 2.1.** A *traintrack (graph)* on a closed surface is an embedded graph with  $C^1$ -smooth edges (*branches*) such that

- each vertex has degree either two or three,

- at each vertex locally  $C^1$ -diffeomorphic to one of the figures in Figure 1, so that is is a union of  $C^1$ -smooth segment, and
- it complementary regions are not disks, mono-gons, or bi-gons.

Vertices of a traintrack are called *switches*.

A *weighted* traintrack is a traintrack with an assignment of non-negative real numbers to each branch, such that at any switch, they satisfy an *switch* condition:  $w_1 + w_2 = w_3$  or  $w_4 = w_5$  as in Figure 1.

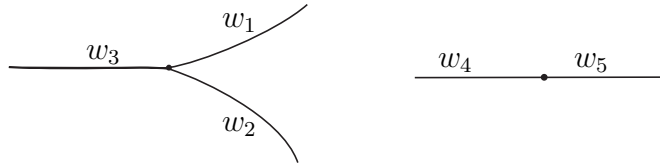


FIGURE 1.

A weighted traintrack corresponds to a measured lamination on the surface. Then we say that the measure lamination is *carried* by the traintrack. In the case when the weights are all integers, a weighted traintrack corresponds to a multiloop (of integer weight).

A traintrack is *maximal* if its complementary regions are  $C^1$ -diffeomorphic to a triangle. Then all possible weights satisfying the switch conditions on a fixed maximal traintrack is a convex cone in a Euclidean space of the dimension equal to the number of the branches. Then every measured lamination on the surface is carried by one of finitely many maximal traintracks. Thus the space of measured laminations  $\mathcal{ML}$  on  $S$  enjoys a structure of a piecewise linear manifold.

**2.2.1. Fat traintracks.** Given a rectangle  $R$ , in this paper, we always choose a pair of its opposite edges to be horizontal and the other pair to be vertical. In addition we decide left and right vertical edges of it.

A *fat traintrack*  $T$  on a surface  $S$  is a union of rectangles  $R$  glued along its vertical edges so that a neighborhood of each vertical edge look like one of the pictures in Figure 2: A vertical edge is either a union of two vertical edges or is identified to another vertical edge. In the former case, a *switch* point decomposes the long vertical edge into the two short edges. We often denote a fat traintrack as  $T = \{R_i\}$ , where  $R_i$  are rectangles. Note that every fat traintrack converges to a traintrack graph by a  $C^1$ -smooth isotopy that collapses rectangles to edges so that the heights of rectangles limit to zero. Thus, similarly, each rectangle  $R$  of the traintrack is called a *branch*. If a vertical edge of a branch is equal to

or contained in a vertical edge of another branch, then we say that those two branches are *adjacent*. Then the *boundary*  $\partial T$  of  $T$  is the union of horizontal edges of rectangles, and it is a smooth except at switch points. Let  $|T|$  denote the union of the rectangles of  $T$ .

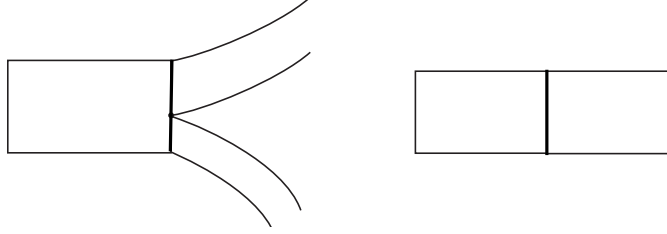


FIGURE 2. Vertical edges separating adjacent branches

A measured lamination  $M$  on a surface is *carried* by a fat traintrack  $T$  if

- $M$  is contained in  $|T|$ ,
- $M$  is transversal to vertical edges of rectangles, and
- for each branch  $R$  of  $T$ ,  $M \cap R$  is a union of arcs connecting opposite vertical edges of  $R$ .

Then the weight of  $M$  on a branch  $R$  of  $T$ , is the transversal measure, given by  $M$ , of a vertical edge of  $R$ .

**2.3. Approximating measured laminations by multiloops.** Let  $L$  be a measured lamination carried by a maximal traintrack  $T$  with  $n$  branches with (positive real) weights  $(w_1, w_2, \dots, w_n)$ , which satisfies the switch conditions.

By scaling a transversal measure by  $t > 0$ , we obtain a ray  $tL = (tw_1, tw_2, \dots, tw_n)$  in  $\mathcal{ML}$ . Then every point on this ray is approximated by a multiloop carried by the traintrack so that the differences of weights is uniformly bounded:

**Lemma 2.2** (Lemma 6.16 of [Gup]). *There exists a constant  $D > 0$ , such that, for every  $t > 0$ , there is a multiloop  $M_t$  carried by  $T$  such that, letting  $(m_1(t), \dots, m_n(t)) \in \mathbb{Z}_{>0}^n$  be the weights on  $T$  representing  $M_t$ ,*

$$(1) \quad -D < 2\pi m_i(t) - tw_i < D$$

for each  $1 \leq i \leq n$ .

**2.4. Admissible loops and projective grafting.** Hyperbolic structures on surfaces are in particular projective structures. First we describe a grafting of a hyperbolic structure along a measured lamination.

Let  $\exp: \mathbb{C} \rightarrow \mathbb{C} \setminus \{0\}$  be the exponential map. Then, for  $\theta > 0$ , a *crescent* of angle  $\theta$  is a projective structure on  $\mathbb{R} \times [0, \theta]$  given by restricting  $\exp$ . Then if

$\theta < 2\pi$ , the crescent of angle  $\theta$  is isomorphic to the region on  $\hat{\mathbb{C}}$  bounded by two embedded circular arcs with common endpoints (*vertices*), intersecting at angle  $\theta$ . If  $\theta > 2\pi$ , the boundary of a crescent region similarly corresponds to such circular arcs but the crescent is immersed in  $\hat{\mathbb{C}}$ . The horizontal translations of  $\mathbb{R}^2$  restrict projective isomorphisms of the crescent  $\mathbb{R} \times [0, \theta]$ . Then given  $a > 0$ , there is a unique infinite cyclic group of such translations, so that the generator corresponds to a hyperbolic element in  $\mathrm{PSL}_2(\mathbb{C})$  of translation length  $a$ . Then the quotient of the crescent by the cyclic group is a projective structure on a cylinder, and its boundary circles are isomorphic to a geodesic loop on a hyperbolic surface whose length is  $a$ .

Let  $\ell$  be a geodesic loop on a hyperbolic surface  $\tau$ . Then, for any  $\theta > 0$ , then there is a corresponding projective cylinder obtained by quotienting a crescent of angle  $\theta$  so that its boundary circles are isomorphic to  $\ell$ . Thus we can insert the cylinder along the loop  $\ell$ , and we obtain new projective structure homeomorphic to  $\tau$ . This operation is a *grafting* of  $\tau$  along  $(\ell, \theta) \in \mathcal{ML}$ . If there are disjoint weighted closed geodesics, we can simultaneously intersect such cylinders of corresponding heights. A general measured lamination can be approximated by weighted multiloops. Thus grafting of  $\tau$  along a measured lamination is as a limit of grafting along multiloops. We refer to [KT92, EM87] for details.

Let  $S$  be a closed orientable surface of genus at least two. Then any projective structure  $C$  on  $S$  is obtained by grafting a hyperbolic surface  $\tau$  along a measured lamination  $L$ , and the choices of  $\tau$  and  $L$  are unique ([KT92, KP94]). Thus we denote as  $C = Gr_L(\tau)$ . We refer to  $(\tau, L)$  as the *Thurston coordinates* of the projective structure  $C$ , and write  $C \cong (\tau, L)$ . Recalling the projection  $\pi: \mathcal{P} \rightarrow \mathcal{T}$ , let  $gr_L(\tau) = \pi \circ Gr_L(\tau)$ .

Note that above we graft along a measured lamination  $L$  appearing circular on  $\hat{\mathbb{C}}$ . In addition  $\mathrm{hol} \, \mathrm{Gr}_{tL}(C) \in \chi$  changes when  $t > 0$  changes. Next we consider a variation of grafting along a multiloop on a (general) projective surface, which does not change holonomy.

Given a projective structure  $C = (f, \rho)$ , a loop  $\ell$  on  $C$  is *admissible* if  $\rho(\ell)$  is loxodromic and a lift of  $\ell$  to the universal cover embeds in  $\hat{\mathbb{C}}$  by the developing map  $f$ . An *admissible multiloop* is a union of a disjoint collection of such admissible loops.

If  $\ell$  is admissible, the loxodromic  $\rho(\ell)$  generates a infinite cyclic group in  $\mathrm{PSL}_2(\mathbb{C})$ . Then, by a Möbius transformation, we can regard 0 and  $\infty$  to be the fixed points of the cyclic group, so that  $f$  embeds  $\tilde{\ell}$  into  $\mathbb{C} \setminus \{0\}$ . By quotienting  $\mathbb{C} \setminus \{0\}$  by the cyclic group, we obtain a projective structure  $T_{\rho(\ell)}$  on a torus (*Hopf torus*). Then  $\ell$  is isomorphically embedded into  $T_{\rho(\ell)}$ . Similarly we can transform a projective structure  $C$  to a new projective structure  $\mathrm{Gr}_\ell(C)$  by inserting the cylinder  $T_{\rho(\ell)} \setminus \ell$  along  $\ell$ . In particular such grafting is often called  *$2\pi$ -grafting*. It turns out that  $\mathrm{Gr}_\ell(C)$  has also holonomy  $\rho$  (see [Gol87] and [Kap01]). If there

are disjoint admissible loops on  $C$ , we can graft  $C$  simultaneously along all loops and obtain a new projective structure, whose holonomy is indeed  $\rho$ . Similarly let  $\text{Gr}_M(C)$  denote the resulting projective structure, where  $M$  is the union of the admissible loops.

*2.4.1. Collapsing maps, pleated surfaces, and Thurston metric.* See [KT92, KP94] (c.f. [Baba]). Let  $C \cong (\tau, L)$  be a projective structure on  $S$  in Thurston coordinates. Then there is a corresponding measured lamination  $\mathcal{L}$  on  $C$  consisting of circular leaves. Moreover there is a natural *collapsing map*  $\kappa: C \rightarrow \tau$  such that  $\kappa$  preserves the marking of the surface and  $\mathcal{L}$  descends to  $L$  by  $\kappa$ . Given a (measured) lamination on a surface, a *stratum* is a leaf of the lamination or a connected component of the complementary region. In particular  $\kappa$  takes each stratum of  $(C, \mathcal{L})$  to a stratum of  $(\tau, L)$ . Let  $(\tilde{C}, \tilde{\mathcal{L}})$  be the universal cover of  $(C, \mathcal{L})$ . Then the developing map  $f$  of  $C$  embeds each stratum of  $(\tilde{C}, \tilde{\mathcal{L}})$  onto a region in  $\hat{\mathbb{C}}$  bounded by circular arcs.

The inverse-image of  $L$  on  $\tau$  is a measured lamination  $\tilde{L}$  on  $\mathbb{H}^2$  invariant under  $\pi_1(S)$ . Then we obtain a *pleated surface*  $\beta: \mathbb{H}^2 \rightarrow \mathbb{H}^3$  bending  $\mathbb{H}^2$  inside  $\mathbb{H}^3$  along  $L$  by the angles corresponding to its transversal measure of  $\tilde{L}$ . Then  $\beta$  is equivariant through  $\text{hol}(C) =: \rho$  (up to an element of  $\text{PSL}_2(\mathbb{C})$ ), as  $f$  is.

Then, letting  $\tilde{\kappa}: \tilde{C} \rightarrow \mathbb{H}^2$  be the lift of  $\kappa: C \rightarrow \tau$  to a collapsing map between the universal covers, indeed  $\tilde{\kappa}$  relates the developing map  $f: \tilde{C} \rightarrow \hat{\mathbb{C}}$  and the pleated surface  $\beta: \mathbb{H}^2 \rightarrow \mathbb{H}^3$ : Namely, if  $\kappa$  diffeomorphically takes a stratum  $\mathcal{P}$  of  $(\tilde{C}, \tilde{\mathcal{L}})$  to a stratum  $P$  of  $(\mathbb{H}^2, \tilde{L})$ , then, recalling that  $f$  embeds  $\mathcal{P}$  into  $\hat{\mathbb{C}}$ ,  $f(\mathcal{P})$  is diffeomorphic to  $\beta(P)$  by the nearest point projection to a hyperbolic plane in  $\mathbb{H}^3$  containing  $\beta(P)$ .

The Thurston metric on  $C$  is a natural Euclidean/hyperbolic metric such that the restriction of  $\kappa$  to each stratum  $P$  of  $(C, L)$  is an isometric embedding. Let  $M$  be the sublamination of  $L$  consisting the closed leaves  $m_1, \dots, m_k$  of  $L$ , so that  $M = m_1 \sqcup \dots \sqcup m_k$ . Then  $\kappa^{-1}(m_i)$  is a cylinder foliated by parallel closed leaves of  $\mathcal{L}$  that are diffeomorphic to  $m_i$  by  $\kappa$  for each  $i \in 1, \dots, k$ . Then  $\kappa^{-1}(m_i)$  enjoys a natural Euclidean metric so that its circumference is the length of  $m_i$  and its height is the weight of  $m_i$ . Thus  $\kappa^{-1}(m_1), \dots, \kappa^{-1}(m_k)$  are disjoint Euclidean cylinders in  $C$ .

On the other hand,  $\kappa$  embeds  $\kappa^{-1}(M)$  onto  $\tau \setminus M$ . Then the Thurston metric on  $C \setminus \kappa^{-1}(M)$  is hyperbolic so that  $\kappa$  is an isometric embedding. Then the Euclidean metric and hyperbolic metric coincide along the boundary of the supporting sub-surfaces and they yield a Euclidean/hyperbolic metric on  $C$ .



## 3. FAT TRAINTRACKS AND LAMINATIONS

Let  $X, Y$  be metric spaces with distances  $d_X, d_Y$  and  $\phi: X \rightarrow Y$  be a continuous map. Then,  $\phi$  is  $A$ -bilipschitz for  $A \geq 1$ , if

$$\frac{1}{A}d_X(p, q) < d_Y(\phi(p), \phi(q)) < Ad_X(p, q),$$

for all  $a, b \in X$ . Given  $B \geq 0$ ,  $\phi$  is a  $B$ -rough isometry if

$$d_X(p, q) - B < d_Y(\phi(p), \phi(q)) < d_X(p, q) + B.$$

For  $A \geq 1, B > 0$ ,  $\phi$  is an  $(A, B)$ -bilipschitz map  $\phi$  is an  $A$ -bilipschitz map and also a  $B$ -rough isometry. In addition, throughout this paper,  $(A, B)$ -bilipschitz maps are piecewise differentiable.

**Definition 3.1** (Nearly straight traintracks). For  $\epsilon > 0$  and  $w > 0$ , a fat traintrack on a hyperbolic surface is  $(\epsilon, w)$ -nearly straight, if each branch is smoothly  $(1 + \epsilon)$ -rough isometric to some euclidean rectangle of width at least  $w$  and at each switch point, the angle between horizontal edges meeting at a switch point is less than  $\epsilon$ .

Here the *width* of a rectangle is the distance between the vertical edges. Note that, given  $w > 0$ , if  $\epsilon > 0$  is sufficiently small, then each  $(\epsilon, w)$ -nearly straight traintrack almost looks like a geodesic lamination.

**Definition 3.2** (Angles between laminations; [Baba]). Let  $\ell$  and  $n$  be simple geodesics on a hyperbolic surface  $\tau$ . If  $\ell$  and  $n$  intersect at a point  $p$ , let  $\angle_p(\ell, n)$  be the angle, taking a value in  $[0, \pi/2]$ , between  $\ell$  and  $n$  at  $p$ . More generally let  $\lambda$  and  $\nu$  be geodesic laminations on  $\tau$ . The angle  $\angle_\tau(\lambda, \nu)$  be the supremum of  $\angle_p(\ell_p, n_p)$  over all points  $p$  of the intersection  $|\lambda| \cap |\nu|$ , where  $\ell_p$  and  $n_p$  are the leaves of  $\lambda$  and  $\nu$  intersecting at  $p$ .

To determine angles defined in Definition 3.2, we always take geodesic representatives of loops and laminations even if they are given as topological loops or laminations.

Then it follows from basic hyperbolic geometry:

**Lemma 3.3.** *Let  $T$  be a fat traintrack on a hyperbolic surface. Let  $\mu, \nu$  be laminations carried by  $T$ . For any  $\epsilon > 0$ , there is  $\delta > 0$  such that if  $T$  is  $(\delta, \epsilon)$ -nearly straight then  $\angle_\tau(\mu, \nu) < \epsilon$ .*

Let  $\tau$  be a closed hyperbolic surface. Consider an essential loop on  $\tau$  of shortest length. Let  $K$  be the one-third of its length. Let  $\lambda$  be a geodesic lamination on  $\tau$ . Then

**Lemma 3.4** (c.f. [Baba]). *For every  $\epsilon > 0$ , there is an  $(\epsilon, K)$ -nearly straight fat traintrack  $T$  carrying  $\lambda$  on  $\tau$ .*

Then we have

**Lemma 3.5.** *Let  $\tau$  be a hyperbolic surface. Consider the shortest geodesic loop on  $\tau$  and let  $K > 0$  be the one third of its length. Then for every measured lamination  $L$  on  $\tau$  and  $\epsilon > 0$ , there is a neighborhood  $U$  of  $L$  in  $\mathcal{ML}$ , such that for every  $M$  in  $U$ , there is an  $(\epsilon, K)$ -nearly straight traintrack on  $\tau$  carrying  $L$  and  $M$ .*

*Proof.* Suppose, to the contrary, that there is no such a neighborhood  $U$ . There is a sequence of measured laminations  $M_i$  that converges to  $L$  such that, for each  $i$ , there is no  $(\epsilon, K)$ -nearly straight traintrack on  $\tau$  carrying both  $L$  and  $M_i$ . Then, since the space  $\mathcal{GL}$  of geodesic laminations on  $S$  is compact, we can in addition assume that  $|M_i|$  converges to a geodesic lamination  $\lambda$  by taking a subsequence. Then since  $M_i \rightarrow L$ , thus  $\angle_\tau(L, \lambda) = 0$ . Therefore  $\lambda \cup |L|$  is a geodesic lamination on  $\tau$ . Thus, by Lemma 3.4, there is an  $(\epsilon, K)$ -nearly straight traintrack  $T$  carrying  $\lambda \cup |L|$ . Clearly  $T$  carries  $L$ . In addition, since  $M_i$  converges to  $L$ , thus  $M_i$  is carried by  $T$  for large enough  $i$ —this is a contradiction.  $\square$

**3.1. Fat traintracks on projective surfaces.** Given a projective structure  $C \cong (\tau, L)$  in Thurston coordinates, there is a nearly straight traintrack  $T$  on  $\tau$  carrying  $L$  by Lemma 3.4. In this section we construct a corresponding “nearly circular” traintrack on  $C$ , as used in [Baba, Babb].

**Definition 3.6.** A *round cylinder* is a cylinder  $\mathcal{A}$  embedded in the Riemann sphere  $\hat{\mathbb{C}}$  bounded by round circles. Recalling that  $\partial\mathbb{H}^3 = \hat{\mathbb{C}}$ , there is, in  $\mathbb{H}^3$ , a unique geodesic orthogonal to the hyperbolic planes bounded by boundary circles of  $\mathcal{A}$ —we call this geodesic the *axis* of  $\mathcal{A}$ . Then this axis contains a geodesic segment that connects the hyperbolic planes, which we call the *core* of  $\mathcal{A}$ . Then  $\mathcal{A}$  is foliated by round circles bounding disjoint copies hyperbolic planes orthogonal to the axis (*vertical foliations of  $\mathcal{A}$* ).

*Remark.* Let  $m$  be the modulus of  $\mathcal{A}$ . Then  $2\pi m$  is the distance between the hyperbolic planes bounded by boundary circles of  $\mathcal{A}$ .

**Definition 3.7** (Canonical metrics on round cylinders). Let  $\mathcal{A}$  be a round cylinder on  $\hat{\mathbb{C}}$  and let  $g$  be the axis of  $\mathcal{A}$ . Then  $\hat{\mathbb{C}}$  minus the end points of  $g$  is biholomorphic to an infinite Euclidean cylinder. Its universal cover is biholomorphic to  $\mathbb{C}$ , which is biholomorphic to the Euclidean plane. Thus this Euclidean metric induces a Euclidean metric on  $\mathcal{A}$ . Since its Euclidean metric is unique up to scaling, we normalize it so that each round circle of the vertical foliation of  $\mathcal{A}$  has length  $2\pi$ .

An arc  $\alpha$  is *supported* on a round cylinder  $\mathcal{A}$  on  $\hat{\mathbb{C}}$  if it is properly embedded in  $\mathcal{A}$  and transversal to the vertical foliation of  $\mathcal{A}$ , so that it connects the boundary components of  $\mathcal{A}$ .

Let  $R$  be a projective structure on rectangle. Then  $R$  is supported on a round cylinder  $\mathcal{A}$  on  $\hat{\mathbb{C}}$  if  $dev(R)$  is into  $\mathcal{A}$ , its horizontal edges are supported on  $\mathcal{A}$  and its vertical edges immerses into boundary circles of  $\mathcal{A}$  (Figure 3). In particular  $R$

is a *circular rectangle* supported on  $\mathcal{A}$  if its horizontal edges are orthogonal to the vertical foliations of  $\mathcal{A}$ .

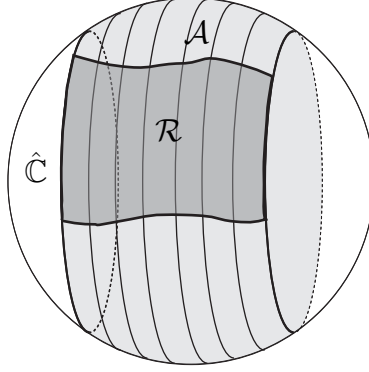


FIGURE 3. A rectangle supported on a round cylinder (embedded case)

**Definition 3.8** (Foliation and metric on rectangles). Let  $R$  be a projective structure on a rectangle supported on a round cylinder  $\mathcal{A}$ . Let  $\mathcal{F}$  be the canonical foliation of  $\mathcal{A}$  by round circles. Let  $e_1, e_3$  be its horizontal edges and  $e_2, e_4$  its vertical edges, so that, via  $\text{dev}(R)$ ,  $e_1$  and  $e_2$  immerse into different boundary circles of  $\mathcal{A}$  and  $e_2$  and  $e_4$  are transversal to  $\mathcal{F}$ . Then we can pull back  $\mathcal{F}$  to a foliation of  $R$  (*vertical foliation*). Similarly  $R$  enjoys a natural Euclidean metric by pulling back the Euclidean metric on  $\mathcal{A}$  via  $\text{dev}(R)$ .

Note that if a rectangle  $R$  is supported on a round cylinder, then with the canonical Euclidean metric,  $R$  isometrically embeds in  $\mathbb{R}^2$  (so that the leaves of its vertical foliation maps to vertical lines).

**Definition 3.9** (Nearly circular arcs). Let  $\alpha$  be an arc supported on a round cylinder  $\mathcal{A}$ , and let  $\epsilon > 0$ . Then  $\alpha$  is  $\epsilon$ -nearly circular if  $\alpha$  is contained in a circular rectangle of height at most  $\epsilon$  and it intersects each vertical leaf of  $\mathcal{A}$  at an angle  $\epsilon$ -close to  $\pi/2$ .

A fat traintrack  $\mathcal{T} = \{\mathcal{R}_i\}$  on a projective surface is *supported* on round cylinders if each branch  $\mathcal{R}_i$  is supported on a round cylinder on  $\hat{\mathbb{C}}$  by the developing map. Then

**Theorem 3.10** ([Baba] Lemma 7.2.). *Let  $\mathcal{T}$  be a fat traintrack on a projective surface. If  $\mathcal{T}$  is supported on round cylinders, then every loop carried by  $\mathcal{T}$  is admissible.*

**Definition 3.11** (Nearly circular rectangles). Let  $R$  be a rectangular projective surface supported on a round cylinder  $\mathcal{A}$ . The rectangle  $R$  is *circular* if its horizontal edges are circular arcs on  $\mathcal{A}$  that are orthogonal to  $\mathcal{F}$ .

More generally, for  $\epsilon > 0$  and  $K > 0$ , the rectangle  $R$  is  $(\epsilon, K)$ -*nearly circular*, if

- the core of  $\mathcal{A}$  has length at least  $K$ , and
- each horizontal edge of  $R$  is  $\epsilon$ -circular.

Note that, by the second condition, if  $R$  is an  $(\epsilon, K)$ -nearly circular rectangle, then the Euclidean lengths of vertical leaves of  $R$  are  $2\epsilon$ -close.

**Definition 3.12** (Nearly concentric cylinders). Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be round cylinders on  $\hat{\mathbb{C}}$ . Let  $c_1$  and  $c_2$  be their cores. Suppose that  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are *adjacent* along a loop  $\ell$ , i.e.  $\mathcal{A}_1$  and  $\mathcal{A}_2$  have disjoint interiors and share a boundary loop  $\ell$ . Then the hyperbolic plane bounded by  $\ell$  contains an endpoint of  $c_1$  and an endpoint of  $c_2$ . Then  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are *concentric* if  $c_1 = c_2$ . More generally, for  $\epsilon > 0$ ,  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are  $\epsilon$ -*nearly concentric*, if the endpoints are  $\epsilon$ -close in the hyperbolic plane.

**Definition 3.13** (Nearly-circular traintrack). A fat traintrack  $\mathcal{T}$  on a projective surface is  $(\epsilon, K)$ -nearly circular if each branch of  $\mathcal{T}$  is  $(\epsilon, K)$ -nearly circular and round cylinders supporting adjacent branches of  $\mathcal{T}$  are  $\epsilon$ -nearly concentric.

3.1.1. *Existence of nearly circular traintracks.* Let  $C \cong (\tau, L)$  be a projective structure on  $S$ . Let  $\lambda$  be a geodesic lamination containing  $|L|$ . Let  $K > 0$  be a constant that does not exceed the one-third of the length of shortest leaf of  $\lambda$ . (In the case that  $\lambda$  contains no closed leaf, fix arbitrary  $K > 0$ .) Then, for every  $\epsilon > 0$  there is an  $(\epsilon, K)$ -nearly straight fat traintrack  $T_{\epsilon, K}$  on  $\tau$  carrying  $\lambda$  (Lemma 3.4). In the following discussion, by “sufficiently straight  $T$ ” we mean an  $(\epsilon, K)$ -nearly straight traintrack  $T$  with sufficiently small  $\epsilon > 0$  while  $K$  is fixed by  $C$ .

Let  $\kappa: C \rightarrow \tau$  be the collapsing map. Let  $\mathcal{L}$  be the measured lamination on  $C$  that descends to  $L$  via  $\kappa$ . We say that a fat traintrack on  $\mathcal{T}$  on  $C$  *descends* to  $T$ , if  $\kappa(|\mathcal{T}|) = |T|$  as subsets of  $\tau$ , and  $\kappa$  bijectively takes branches  $\mathcal{T}$  to branches  $T$  up to a small perturbation of vertical edges. Then

**Theorem 3.14.** *For every  $\epsilon > 0$  and  $K > 0$ , there is  $\delta > 0$  such that given a projective structure  $C \cong (\tau, L)$  on  $S$  and a  $(\delta, K)$ -nearly straight fat traintrack  $T = \{R_i\}$  on  $\tau$  carrying  $L$ , then there is a fat traintrack  $\mathcal{T} = \{\mathcal{R}_i\}$  on  $C$  that descends to  $T$ , up to  $\epsilon$ -small isotopy of vertical edges, such that*

- (1) *if  $R$  and  $\mathcal{R}$  are corresponding branches of  $T$  of  $\mathcal{T}$ , respectively, then*
  - (a)  *$\mathcal{R}$  is supported on a round cylinder on  $\hat{\mathbb{C}}$  and the core of the round cylinder has length  $\epsilon$ -close to the width of the branch  $R$ , and*
  - (b) *each vertical leaf of  $\mathcal{R}$  has length  $\epsilon$ -close to the weight of  $L$  on  $R$ .*
- (2)  *$\mathcal{T}$  is  $(\epsilon, K)$ -nearly circular.*

In [Baba], the first author obtained a similar statement simultaneously for a certain family of projective structures with fixed holonomy. Theorem 3.14 essentially follows from arguments in [Baba, §7.3]. We here outline how to apply the arguments to our setting.

*Outline of the proof.* Let  $\kappa: C \rightarrow \tau$  be the collapsing map. Let  $\tilde{\kappa}: \tilde{C} \rightarrow \mathbb{H}^2$  be its lift to the collapsing map between their universal covers. Let  $\tilde{T}$  and  $\tilde{\mathcal{T}}$  be the lifts of  $T$  to  $\mathbb{H}^2$  and  $\mathcal{T}$  to  $\tilde{C}$ , respectively. Let  $\beta: \mathbb{H}^2 \rightarrow \mathbb{H}^3$  be the pleated surface induced by  $(\tau, L)$ . For each switch point  $p$  of  $\tilde{T}$ , since  $p$  is disjoint from  $\tilde{L}$ , there is a unique point  $\mathbf{p} \in \tilde{C}$  mapping to  $p$  by  $\tilde{\kappa}$ . Then, for every  $\epsilon > 0$ , if  $T$  is sufficiently straight, then, we can pick a round circle  $c_p$  on  $\hat{C}$  for each switch point  $p$  of  $\tilde{T}$  such that

- (1)  $\mathbf{p}$  is contained in  $c_p$ ,
- (2) the hyperbolic plane  $H_p$  in  $\mathbb{H}^3$  bounded by  $c_p$  contains  $\beta(p)$ ,
- (3) if a stratum  $P$  of  $(\mathbb{H}^2, \tilde{L})$  intersects a vertical edge of  $\tilde{T}$  containing  $p$ , then  $\beta(P)$  intersects  $H_p$  transversally at an angle  $\epsilon$ -close to  $\pi/2$ , and
- (4) the choice of  $c_p$ 's is  $\rho$ -equivariant.

Such a family of round circles  $\{c_p\}$  can be constructed, since  $T$  is nearly straight, using the relation between  $dev(C)$  and  $\beta$ . Recall, if  $\mathcal{P}$  is the stratum of  $(\tilde{C}, \tilde{L})$  corresponding to  $P$ , that  $dev(C)$  embeds  $\mathcal{P}$  into  $\hat{C}$ . Then, by (3), if  $\mathcal{P}$  is a circular arc, then  $\mathcal{P}$  intersects  $c_p$  in a single point at an angle  $\epsilon$ -close to  $\pi/2$ . Similarly, if  $\mathcal{P}$  has interior, then  $\mathcal{P}$  intersects  $c_p$  in a circular arc *nearly orthogonally*, that is, exactly two circular arcs of  $\partial\mathcal{P}$  intersect  $c_p$  at angles  $\epsilon$ -close to  $\pi/2$ . Since each branch of  $T$  has width at least  $K$ , if  $\epsilon > 0$  is sufficiently small, then for each branch  $R$  of  $\tilde{T}$ , the round circles corresponding to the switch points in vertical edges are disjoint by (3). Let  $\mathcal{A}_R$  be the round cylinder bounded by the round circles. Then  $\beta(R)$  is  $\epsilon$ -close to the core of  $\mathcal{A}_R$ . Then by (4), this correspondence between branches  $R$  of  $\tilde{T}$  and round cylinders  $\mathcal{A}_R$  is  $\rho$ -equivariant. Moreover, if  $T$  is sufficiently straight,

- (1) if  $R_1$  and  $R_2$  are adjacent branches of  $\tilde{T}$ , then corresponding round cylinders  $\mathcal{A}_{R_1}$  and  $\mathcal{A}_{R_2}$  are adjacent,
- (2) the core of  $\mathcal{A}_R$  has length at least  $K/2$ ,
- (3) if  $P$  is a stratum of  $(\mathbb{H}^2, \tilde{L})$  intersecting a branch  $R$  of  $\tilde{T}$ , then letting  $\mathcal{P}$  be the corresponding stratum of  $(\tilde{C}, \tilde{L})$ ,  $\mathcal{P}$  is  $\epsilon$ -nearly orthogonal to  $\mathcal{A}_R$ , and  $\beta(P)$  is  $\epsilon$ -nearly orthogonal to hyperbolic planes bounded by  $\mathcal{A}_R$ .

Next we shall decompose a subset  $\tilde{\kappa}^{-1}(|\tilde{T}|)$  of  $\tilde{C}$  into rectangles, producing a traintrack supported on the round cylinders  $\mathcal{R}_R$ . Suppose that a stratum  $P$  of  $(\mathbb{H}^2, \tilde{L})$  is disjoint from horizontal edges of  $R$ . Then, if  $P$  is a geodesic,  $P \cap R$  is a geodesic segment, and otherwise  $P$  has positive area and  $P \cap R$  is a rectangle bounded by vertical edges of  $R$  and two leaves of  $\tilde{L}$ . Let  $f = dev(C)$  and let  $\mathcal{P} = f^{-1}(\mathcal{A}_R) \cap \tilde{\kappa}^{-1}(P)$ . Then, if  $T$  is sufficiently straight, then  $\mathcal{P}$  is an  $(\epsilon, K)$ -nearly circular rectangle supported on  $\mathcal{A}_R$ . If the boundary of  $P$  contains a horizontal edge of  $R$ , then  $P \cap R$  is a rectangle bounded by vertical edges of  $R$ , a leaf of  $L$  and a smooth segment in  $\partial\tilde{T}$ . Then let  $\mathcal{P}$  be the corresponding component of  $\kappa^{-1}(P) \cap f^{-1}(\mathcal{A}_R)$  that is a rectangle supported on  $\mathcal{A}_R$ . Let  $\mathcal{R}_R$

be the union of all such  $\mathcal{P}$  over all strata  $P$  of  $(\mathbb{H}^2, \tilde{L})$  intersecting  $R$ . Then  $\mathcal{R}_R$  is a rectangle supported on  $\mathcal{R}$ . Then (3) implies that  $\tilde{\kappa}$  takes  $\mathcal{R}_R$  to  $R$  up to an  $\epsilon$ -small perturbation of vertical edges.

Then, by applying this construction to all branches  $R$  of  $\tilde{T}$ , we see that  $\tilde{\kappa}^{-1}(|\tilde{T}|)$  decomposes into such rectangles  $\mathcal{R}_R$  supported on corresponding round cylinders  $\mathcal{A}_R$ . Then we obtain this structure of a traintrack on  $\tilde{\kappa}^{-1}(|\tilde{T}|)$  that is diffeomorphic to  $\tilde{T}$  and it carries  $\tilde{\mathcal{L}}$ . Since it is invariant under  $\pi_1(S)$ , we obtained a traintrack  $\mathcal{T}$  on  $C$  carrying  $\tilde{\mathcal{L}}$ .

From this construction, in Theorem 3.14, we have in addition

**Proposition 3.15.** *(1) If  $\ell$  is a leaf of  $\tilde{\mathcal{L}}$  passing a branch  $\mathcal{R}$  of  $\tilde{\mathcal{T}}$ , then  $\ell \cap \mathcal{R}$  is a  $\epsilon$ -circular arc supported on  $\mathcal{A}$ , and  $\beta \circ \tilde{\kappa}$  takes the arc  $\ell \cap \mathcal{R}$  to a curve  $\epsilon$ -close to the core of  $\mathcal{A}$ .*  
*(2) Similarly if  $\ell$  is a smooth segment of  $\partial\tilde{\mathcal{T}}$  intersecting  $\mathcal{R}$ , then,  $\ell \cap \mathcal{A}$  is an  $\epsilon$ -nearly circular arc supported on  $\mathcal{A}$ , and  $\beta \circ \tilde{\kappa}$  takes the arc  $\ell \cap \mathcal{R}$  to a curve  $\epsilon$ -close to the core of  $\mathcal{A}$ .*

3.14

#### 4. THE PROOF OF THEOREM 1.3

Let  $C \cong (\tau, L)$  be a projective structure on  $S$  in Thurston coordinates. We prove

**Theorem 1.3.** For every  $\epsilon > 0$ , if  $M$  is sufficiently close to  $L$  in  $\mathcal{ML}$ , then for every sufficiently large  $t > 0$ , there is an admissible loop  $N$  on  $C$ , such that,

$$d(gr_N(C), gr_{tM}(\tau)) < \epsilon.$$

Consider the shortest closed geodesic loop on  $\tau$ , and let  $K$  be the one-third of its length. Then by Lemma 3.5, for every  $\epsilon > 0$ , if  $M$  is a measured geodesic lamination on  $S$  and it is sufficiently close to  $L$ , then there is an  $(\epsilon, K)$ -nearly straight traintrack  $T$  on  $\tau$  carrying both  $L$  and  $M$ . As before, by “sufficiently straight  $T$ ”, we mean that  $\epsilon > 0$  is sufficiently small, while  $K$  is fixed, which is realized by taking  $M$  sufficiently close to  $L$ .

Let  $\mathcal{L}$  be the circular measured lamination on  $C$  that descends to  $L$  by the collapsing map  $\kappa: C \rightarrow \tau$ . If  $M$  is sufficiently close to  $L$ , then by Theorem 3.14, there is in addition an  $(\epsilon, K)$ -nearly circular traintrack  $\mathcal{T}$  on  $C$  carrying  $\mathcal{L}$  such that  $\mathcal{T}$  is admissible and descends to  $T$ , up to  $\epsilon$ -small homotopies of vertical edges.

For  $t \geq 0$ , let

$$C_t = Gr_{tM}(\tau).$$

Since  $T$  is  $(\epsilon, K)$ -nearly straight traintrack carrying  $M$ , if  $M$  is sufficiently close to  $L$ , then let  $\mathbb{T}_t$  be the  $(\epsilon, K)$ -nearly circular traintrack on  $C_t$  that descends to  $T$  (Theorem 3.14). Let  $\{\mathcal{R}_i\}$  denote its branches.

Note that  $\mathcal{T}$  and  $\mathbb{T}_t$  are diffeomorphic to  $T$  as  $C^1$ -smooth traintracks via (perturbations of) collapsing maps. By Lemma 2.2, there is a constant  $D > 0$ , such that for every  $t > 0$  there is a multiloop  $N (= N_t)$  on  $C$  carried by  $\mathcal{T}$  such that the weights of  $N + L$  and  $tM$  are  $D$ -close on each branch.

Since  $N$  is carried by the fat traintrack  $\mathcal{T}$  supported by round cylinders,  $N$  is admissible, by Theorem 3.10. Let

$$C_N = \text{Gr}_N(C).$$

Since  $N$  is carried by  $\mathcal{T}$ , thus

$$\text{Gr}_N(|\mathcal{T}|)$$

is a subsurface of  $C_N$ . Each branch  $\mathcal{R}$  of  $\mathcal{T}$  is an  $(\epsilon, K)$ -nearly circular rectangle supported on a round cylinder. Thus  $N \cap \mathcal{R}$  is an admissible multiarc on  $\mathcal{R}$  (see [Baba]) and the graft of  $\mathcal{R}$  along  $N \cap \mathcal{R}$  is an  $(\epsilon, K)$ -nearly circular rectangle supported on the same cylinder. With the induced Euclidean metric, this grafts increases the length of each vertical leaf by  $2\pi$  times the number of the arcs of  $N \cap \mathcal{R}$ . Then

$$\text{Gr}_N(|\mathcal{T}|)$$

decomposes into the grafts of each branches  $\mathcal{R}$  of  $\mathcal{T}$ , and yields a fat traintrack denoted by  $\mathcal{T}_N$ . Therefore  $\mathcal{T}_N$  is an  $(\epsilon, K)$ -nearly circular traintrack diffeomorphic to  $\mathcal{T}$  such that their corresponding branches are supported on the same round cylinders.

Then there is  $\hat{\kappa}_N: C_N \rightarrow C$  be the natural collapsing map, such that it collapses grafting cylinders onto  $N$  and thus it realizes the inclusion  $C \setminus N \subset C_N$ . Let  $\kappa_N: C_N \rightarrow \tau$  be the composition of the collapsing maps  $\kappa_C$  and  $\hat{\kappa}_N$ . By Theorem 3.14 and the relation between  $\mathcal{T}$  and  $\mathcal{T}_N$ , discussed above, we have

**Proposition 4.1.** *For every  $\epsilon > 0$ , if  $T$  is sufficiently straight, then, for every  $t > 0$ ,  $\mathcal{T}_N$  is an  $(\epsilon, K)$ -nearly circular fat traintrack on  $C_N$ , such that*

- (1)  $\kappa_N: C_N \rightarrow \tau$  descends  $\mathcal{T}_N$  to  $T$  up to a  $(\epsilon)$ -small homotopies of vertical edges,
- (2) if  $R$  is branch of  $T$  and  $\mathcal{R}$  is the corresponding branch of  $\mathcal{T}_N$  supported on the round cylinder  $\mathcal{A}$ , then
  - (a) the length of the core of  $\mathcal{A}$  is  $\epsilon$ -close to the width of  $R$ .
  - (b) each vertical leaf of  $\mathcal{R}$  has Euclidean length  $\epsilon$ -close to the weight of  $L + N$  on the branch  $R$ .

Thus, for every  $\epsilon > 0$ , if  $M$  is sufficiently close to  $L$  in  $\mathcal{ML}$ , then for sufficiently large  $t > 0$ , we shall construct a piecewise-smooth  $(1 + \epsilon)$ -quasiconformal map

from  $C_N$  to  $C_t$  that preserve the marking of the surface; the theorem then follows.

Here is an outline of the construction. ( $\epsilon > 0$  is any fixed number.)

*Step 1.* The complement of the traintracks  $\mathcal{T}_N$  and  $\mathcal{T}_t$  in  $C_N$  and  $C_t$  are hyperbolic with Thurston metric on  $C_N$  and  $C_t$ , and, there is a natural hyperbolic isometry

$$\psi: C_t \setminus \mathcal{T}_t \rightarrow C_N \setminus \mathcal{T}_N.$$

*Step 2.* We next construct a quasiconformal map

$$\phi: \mathcal{T}_N \rightarrow \mathcal{T}_t$$

that has small distortion if  $t > 0$  is sufficiently large. This map is given by natural “linear” maps between corresponding branches with respect appropriate Euclidean metrics obtained by slightly modifying the canonical Euclidean metrics. In particular, there is  $H > 0$ , if  $T$  is sufficiently straight, such that this map is  $(1 + \epsilon, H)$ -bilipschitz for sufficiently large  $t > 0$ .

*Step 3.* We modify  $\phi$  so that it coincides with  $\psi$  along  $\partial\mathcal{T}_N$  so that  $\phi$  and  $\psi$  define a (continuous)  $(1 + \epsilon, H)$ -bilipschitz map  $C_N \rightarrow C_t$ .

Since  $\phi$  is in particular homeomorphism  $\mathcal{T}_N \rightarrow \mathcal{T}_t$ , it restricts to a homeomorphism  $\partial\phi: \partial\mathcal{T}_N \rightarrow \partial\mathcal{T}_t$ . The complements  $C_N \setminus \mathcal{T}_N$  and  $C_t \setminus \mathcal{T}_t$  are identified by the hyperbolic isometry  $\psi$ , which restricts to an isometry from  $\partial\mathcal{T}_N$  to  $\partial\mathcal{T}_t$ . Then  $\partial\phi$  is an  $(1 + \epsilon, \epsilon)$ -bilipschitz map with the complementary hyperbolic metrics. Lastly adjust  $\phi: \mathcal{T}_N \rightarrow \mathcal{T}_t$  near  $\partial|\mathcal{T}_N|$  by postcomposing with an  $(1 + \epsilon, \epsilon)$ -bilipschitz map, so that  $\phi$  and  $\psi$  match up on the boundary  $\partial\mathcal{T}_N$ .

Therefore, if  $T$  is sufficiently straight and  $t > 0$  is sufficiently large, then we have a  $(1 + \epsilon, K)$ -bilipschitz map  $C_N \rightarrow C_t$  given by  $\psi$  and (modified)  $\phi$  with respect to appropriate Euclidean/Hyperbolic metrics.

**4.1. Step 1: Isometry between the complements of traintracks.** Recall that the collapsing maps  $\kappa: C \rightarrow \tau$  and  $\kappa_t: C_t \rightarrow \tau$  take  $|\mathcal{T}|$  and  $|\mathcal{T}_t|$  to the nearly straight traintrack  $|T|$  carrying both measured laminations  $M$  and  $L$ . Thus Thurston metrics on  $C_t \setminus \mathcal{T}_t$  and  $C \setminus \mathcal{T}$  are hyperbolic, and moreover  $C_t \setminus \mathcal{T}_t$  is isometric to  $C \setminus \mathcal{T}$  via  $\kappa^{-1} \circ \kappa_t$ .

Recall the collapsing map  $\hat{\kappa}_N: C_N \rightarrow C$ . Since  $N$  is carried by  $\mathcal{T}$ , thus  $\hat{\kappa}_N$  restricts to an isomorphism from  $C_N \setminus \mathcal{T}_N$  to  $C \setminus \mathcal{T}$  as projective surfaces. Then  $C_N \setminus \mathcal{T}_N$  enjoys a hyperbolic metric obtained by pulling back the hyperbolic metric on  $\tau$  by the composition of collapsing maps,  $\kappa_N = \kappa \circ \hat{\kappa}_N: C_N \rightarrow \tau$ . Then let  $\psi: C_N \setminus \mathcal{T}_N \rightarrow C_t \setminus \mathcal{T}_t$  be the natural hyperbolic isometry give by  $\kappa_t^{-1} \circ \kappa_N$ .



## 4.2. Step 2: Linear map between traintracks.

4.2.1. *Linear maps between rectangles supported on round cylinders.* Let  $\mathcal{A}, \mathcal{A}'$  be round cylinders on  $\hat{\mathbb{C}}$ . Let  $R$  and  $R'$  be projective structures on a rectangle supported on  $\mathcal{A}$  and  $\mathcal{A}'$ , respectively. Then there are canonical Euclidean metrics  $E$  on  $R$  and  $E'$  on  $R'$  by Definition 3.8. Then there is a unique diffeomorphism  $\xi: R \rightarrow R'$  such that

- $\xi$  preserves the left vertical edge, right vertical edge, and horizontal edges,
- $\xi$  takes each vertical leaf of  $R$  onto a vertical leaf of  $R'$  linearly; thus  $\xi$  induces a continuous map  $\xi_*$  in the horizontal coordinate, and
- $\xi_*$  is also linear.

Then

**Lemma 4.2.** *For every  $\epsilon > 0$  and  $K > 0$ , there is  $\delta > 0$ , such that, if*

- (1)  *$R$  and  $R'$  are  $(\delta, K)$ -nearly circular,*
- (2)  *$\delta$  bounds, from above, the difference of the lengths of the cores of  $\mathcal{A}$  and  $\mathcal{A}'$ , and*
- (3) *there are constants  $V, V' > 1/\delta$  with  $-K/2 < V - V' < K/2$ , such that, for all vertical leaves of  $\ell$  and  $\ell'$  of  $R$  and  $R'$ , respectively,*

$$-\delta < \text{length}_E(\ell) - V < \delta$$

$$-\delta < \text{length}_{E'}(\ell') - V' < \delta,$$

*then  $\xi: R \rightarrow R'$  is a smooth  $(1 + \epsilon, K)$ -bilipschitz map.*

*Proof.* First we consider  $\xi$  in the vertical direction. If  $\delta > 0$  is sufficiently small, by (3), for all vertical leaves  $\ell$  of  $R$  and  $\ell'$  of  $R'$  corresponding by  $\xi$ ,  $\text{length}_E(\ell)$  and  $\text{length}_{E'}(\ell')$  are sufficiently large and  $-K < \text{length}(\ell) - \text{length}(\ell') < K$ . Thus  $\xi|_\ell$  is a smooth  $(1 + \epsilon, K)$ -bilipschitz map for all vertical leaf  $\ell$  of  $R$ .

Next we consider in the horizontal direction. Let  $W$  and  $W'$  denote the Euclidean widths of  $R$  and  $R'$ , respectively. Then  $\xi_*: [0, W] \rightarrow [0, W']$  denotes the linear map induced by  $\xi$ . Then, if  $\delta > 0$  is sufficiently small, since  $|W - W'| < \delta$  and  $W, W' > K$ , then  $\xi_*$  is a  $(1 + \epsilon, \epsilon)$ -bilipschitz map.

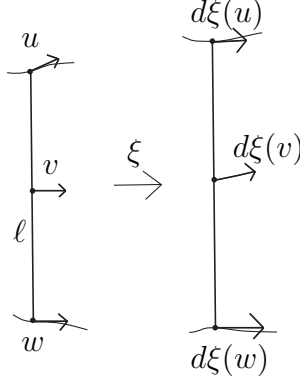


FIGURE 4.

Let  $v$  be a horizontal unit tangent vector at a point  $p$  in  $R$ . Let  $\ell$  be the vertical leaf of  $R$  passing  $p$ . Let  $u$  and  $w$  be the unit tangent vectors tangent to the horizontal edges of  $R$  at the endpoints of  $\ell$ , such that  $u, w$  point the same direction as  $v$  in the horizontal coordinate.

Then if  $\delta > 0$  is sufficiently small, since  $R$  is  $(\delta, K)$ -circular,  $u$  and  $w$  are  $\epsilon$ -nearly horizontal, i.e. the angles between those vectors and appropriate horizontal vectors are less than  $\epsilon$ . Since  $R'$  is  $\delta$ -nearly circular, we can in addition assume that  $d\xi(u)$  and  $d\xi(w)$  are  $\epsilon$ -nearly horizontal. Notice that  $d\xi|_{\ell}$  induces a single affine map from the plane tangent to  $\ell$  to the plane tangent to  $\xi(\ell)$ . Thus  $d\xi(v)$  is  $\epsilon$ -nearly horizontal as well. Therefore, if  $\delta > 0$  is sufficiently small,  $1 - \epsilon < |d\xi(v)| < 1 + \epsilon$  for all horizontal unit tangent vectors  $v$  at points in  $R$ , since  $\xi_*$  is a smooth bilipschitz map with small distortion. In addition, since  $\xi$  is, in the vertical direction, a smooth bilipschitz map with small distortion,  $d\xi$  is in particular  $(1 + \epsilon)$ -quasiconformal (i.e. distorts angles by a factor of at most  $1 + \epsilon$ ).

If  $\delta > 0$  is sufficiently small, since  $\xi$  is a bilipschitz map of small distortion in both vertical and horizontal directions,  $\xi$  is  $(\epsilon + 1)$ -bilipschitz map. We have seen that  $\xi$  is  $K$ -rough isometric in the vertical direction and  $\epsilon$ -isometric in the horizontal direction. Thus if  $\delta > 0$  is sufficiently small, then  $\xi$  is a  $K$ -rough isometry. 4.2

**4.2.2. Linear maps between traintracks.** Recall that we have traintracks  $\mathcal{T}_N$  on  $C_N$  and  $\mathcal{T}_t$  on  $C_t$  that are isomorphic as smooth fat traintracks, and their branches are supported on round cylinders. Then if  $\mathcal{R}_i$  and  $\mathcal{R}_i$  are corresponding branches of  $\mathcal{T}_N$  and  $\mathcal{T}_t$ , then let  $\xi_i: \mathcal{R}_i \rightarrow \mathcal{R}_i$  be the diffeomorphism defined in §4.2.1. Then let  $\xi = \{\xi_i\}_i$ . Although  $\xi_i$ 's typically do not match up along vertical edges for adjacent branches of the traintracks (see §4.2.3), Nonetheless we may regard  $\xi$  as a map  $\mathcal{T}_N \rightarrow \mathcal{T}_t$  that is continuous except along vertical edges.

We check the hypotheses in Lemma 4.2 to  $\xi_i$ . Let  $R$  be a branch of  $T$  on the hyperbolic surface  $\tau$ . Let  $\mathcal{R}_N$  and  $\mathcal{R}_t$  be the branches of  $\mathcal{T}_N$  and  $\mathcal{T}_t$  corresponding to  $R$ . Let  $\mathcal{A}_N$  and  $\mathcal{A}_t$  be the round cylinders supporting  $\mathcal{R}_N$  and  $\mathcal{R}_t$ , respectively.

Then, there is  $K > 0$ , such that, for every  $\delta > 0$ , if  $T$  is sufficiently straight, then  $\mathcal{T}_N$  and  $\mathcal{T}_t$  are  $(\delta, K)$ -nearly circular (Theorem 3.14 (2)), which is Assumption (1) in Lemma 4.2. In addition, by Theorem 3.14 (1 -a), the lengths of the cores of  $\mathcal{A}_N$  and  $\mathcal{A}_t$  are  $\epsilon$ -close, regardless of the choice of  $R$ , which is (2) in Lemma 4.2.

For every  $\delta > 0$ , if  $T$  is sufficiently straight, then by Theorem 3.14 (1 -b) and Proposition 4.1,

- each vertical leaf of each  $\mathcal{R}_N$  has length  $\delta$ -close to the weight of  $L_N$  on  $\mathcal{R}_N$ , and
- each vertical leaf of each  $\mathcal{R}_t$  has length  $\delta$ -close to the weight of  $tN$  on  $\mathcal{R}_t$ .

Recall that  $N = N_t$  is chosen so that the differences of the weights of  $L + N$  and  $M$  on corresponding branches are uniformly bounded by the constant  $D > 0$  independent on  $t$ .

In addition, if  $t > 0$  is sufficiently large, weights of  $tM$  and  $L + N$  are at least  $1/\delta$  on all branches of the traintracks. Thus Lemma 4.2 (3) holds. Therefore, by Lemma 4.2,

**Proposition 4.3.** *There is  $H > 0$  such that, given any  $\epsilon > 0$ , if  $T$  is sufficiently straight, then for all sufficiently large  $t > 0$ ,  $\xi_i: \mathcal{R}_i \rightarrow \mathcal{R}_i$  is a smooth  $(1 + \epsilon, H)$ -bilipschitz map for each branch  $\mathcal{R}_i$  of  $T$  with respect to the Euclidean metrics  $E_N, E_t$  induced by their supporting round cylinders.*

**4.2.3. Matching along adjacent branches.** The map  $\xi: \mathcal{T}_N \rightarrow \mathcal{T}_t$  is defined between corresponding branches (§4.2.2), if  $T$  is sufficiently straight, it is an  $(1 + \epsilon, K)$ -bilipschitz map. Typically, on adjacent branches of  $\mathcal{T}_N$ ,  $\xi$  do not agree along on their common vertical edge: Their supporting round cylinders may not be concentric, and the canonical euclidean metrics on the round cylinders do not agree along their common boundary circle. Nonetheless those round cylinders are nearly concentric, and therefore  $\xi$  is “close” to being continuous along vertical edges. In this section, we modify  $\xi$  so that it is in addition continuous and yet  $(1 + \epsilon, K)$ -bilipschitz.

More generally let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be adjacent round cylinders on  $\hat{\mathbb{C}}$ , which share a boundary circle  $\ell$ . First we make them concentric by applying a natural Möbius transformation to  $\mathcal{A}_2$ , keeping them adjacent: Let  $c_1$  and  $c_2$  be the cores of  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , respectively. Let  $P$  be the hyperbolic plane in  $\mathbb{H}^3$  bounded by  $\ell$ , and let  $p_1$  and  $p_2$  be the end points of  $c_1$  and  $c_2$  on  $P$ . Let  $\gamma \in \text{PSL}_2(\mathbb{C})$  be such that  $\gamma$  restricts to a unique translation of  $P$  with  $\gamma(p_2) = p_1$  along the geodesic

$g$  connecting  $p_1$  and  $p_2$ . Then  $\mathcal{A}_1$  and  $\gamma\mathcal{A}_2$  are still adjacent along  $\ell$ , and  $\mathcal{A}_1$  and  $\gamma\mathcal{A}_2$  are indeed concentric. Note that  $\gamma$ , being a Möbius transformation, preserves the canonical Euclidean metric on  $\mathcal{A}_2$ .

Let  $\mathcal{A}'_2 = \gamma\mathcal{A}_2 \subset \hat{\mathbb{C}}$ . Let  $c'_2 = \gamma c_2 \subset \mathbb{H}^3$ , the core of  $\mathcal{A}'_2$ . Next we shall modify  $\gamma$  by post-composing with a piecewise differentiable homeomorphism  $\eta: \mathcal{A}'_2 \rightarrow \mathcal{A}'_2$  such that  $\eta \circ \gamma$  restricts to the identity map on  $\ell$ . Let  $w$  be the length of  $c'_2$ . Then parametrize  $c'_2$  by arclength as  $c'_2: [-\frac{w}{2}, \frac{w}{2}] \rightarrow \mathbb{H}^3$  so that the endpoint  $c'_2(w/2)$  lies in  $P$ .

Let  $\text{Conv } \mathcal{A}'_2$  denote the convex hull of  $\mathcal{A}'_2$  in  $\mathbb{H}^3$ , which is homeomorphic to a solid cylinder. For each  $t \in [-w/2, w/2]$ , let  $P_t$  be the hyperbolic plane orthogonal to  $c'_2$  at  $c'_2(t)$ . Then  $\{P_t\}$  foliates  $\text{Conv } \mathcal{A}'_2$ . Assigning an orientation on the geodesic  $g$  in the direction from  $p_1$  to  $p_2$ , let  $g_t$  be the oriented geodesic on  $P_t$  obtained by parallel transport of  $g$  along  $c'_2$ . Let  $\eta_t$  be the translation of the hyperbolic plane  $P_t$  along  $g_t$  by  $d_P(p_2, p_1) \cdot \frac{2t}{w}$  for  $t \in [0, \frac{w}{2}]$  and  $\eta_t$  be the identity map for  $t \in [-\frac{w}{2}, 0]$ . This one-parameter family of sometimes  $\eta_t$  yields a piecewise differentiable homeomorphism  $\eta: \text{Conv } \mathcal{A}'_2 \rightarrow \text{Conv } \mathcal{A}'_2$ . Then  $\eta$  restricts a piecewise differentiable homeomorphism  $\eta: \mathcal{A}'_2 \rightarrow \mathcal{A}'_2$ . Indeed the composition  $\eta \circ \gamma: \mathcal{A}_2 \rightarrow \mathcal{A}'_2$  fixes the boundary circle  $\ell$  pointwise and  $\eta \circ \gamma(\mathcal{A}_2) = \mathcal{A}'_2$ , which is concentric to  $\mathcal{A}_1$ .

If a projective structure on a rectangle  $R$  is supported on  $\mathcal{A}_2$ , then by post-composing  $\text{dev}(R)$  with  $\eta \circ \gamma$ , we obtain a new projective structure on  $R$ . Then accordingly we obtained a new Euclidean metric on  $R$ .

Next we see that this new Euclidean metric is very close to the original Euclidean metric if round cylinders are nearly concentric. First, with respect to the canonical Euclidean metric on  $\mathcal{A}'_2$ , we have

**Lemma 4.4.** *For every  $\epsilon > 0$ , there is  $\delta > 0$  such that if the length  $w$  of the core of  $\mathcal{A}'_2$  is more than  $\epsilon$  and the endpoints  $p_1$  and  $p_2$  are  $\delta$ -close, then  $\eta: \mathcal{A}'_2 \rightarrow \mathcal{A}'_2$  is smoothly  $(1 + \epsilon, \epsilon)$ -bilipschitz and  $\epsilon$ -close to the identity map as a piecewise smooth map.*

*Proof.* Let

$$X = \bigcup_{0 \leq t \leq w/2} \partial P_t,$$

be the half of the cylinder  $\mathcal{A}'_2$  where  $\eta$  is supported. Clearly it suffices to show the lemma on  $X$ .

If  $d_P(p_1, p_2)/w$  is sufficiently small, then for each  $t \in [0, w/2]$ ,  $\eta_t$  is smoothly  $\epsilon$ -close to the identity map and in particular  $(1 + \epsilon, \epsilon)$ -bilipschitz. Therefore, since  $\eta$  preserves  $P_t$ 's, if  $\delta > 0$  is sufficiently small, then  $\eta$  is  $\epsilon$ -rough isometry of  $X$ .

Next we show that  $\eta$  is  $(1 + \epsilon)$ -bilipschitz. We saw that  $d\eta$  is  $(1 + \epsilon)$ -bilipschitz in  $\partial P_t$  for each  $t$ . Notice that  $P_t$  are naturally identified. Then, if  $\delta > 0$  is

sufficiently small, then, for each  $t \in [0, \pi/2]$  and  $x \in \partial P_t$ , we have  $\frac{d\eta_t}{dt}(x) < \epsilon$ . Thus, since  $\eta$  preserves  $t$ -coordinates, every horizontal tangent vector on  $X$  stays  $\epsilon$ -almost horizontal and, in terms of its length,  $(1 + \epsilon)$ -bilipschitz. Therefore,  $d\eta$  is bilipschitz in both  $t$  and  $\partial P_t$  directions and thus  $(1 + \epsilon)$ -quasiconformal. Hence, for sufficiently small  $\delta > 0$ ,  $\eta$  is  $(1 + \epsilon)$ -bilipschitz.  $\square$

Given a traintrack  $\mathcal{T}$  supported on round cylinders, we can apply the modification above for all adjacent branches, so that the supporting cylinders are all concentric. Then all supporting cylinders are contained in  $\hat{\mathbb{C}}$  minus two points, the end points of their common axis. Then, letting  $\tilde{\mathcal{T}}$  be the universal cover of  $\mathcal{T}$ , there is a modified developing map from  $\tilde{\mathcal{T}}$  to  $\hat{\mathbb{C}}$  minus two points that is equivariant under a representation of  $\pi_1(\mathcal{T})$  onto a subgroup of  $\mathrm{PSL}_2(\mathbb{C})$  consisting of loxodromic elements fixing the common axis. Thus we have a continuous Euclidean metric on  $\mathcal{T}$  induced by this modified developing map (*modified Euclidean metric*). Then Lemma 4.4 implies

**Proposition 4.5.** *For every  $\epsilon > 0$ , if  $\delta > 0$  is sufficiently small, then given any  $(\delta, \epsilon)$ -nearly circular traintrack supported on round cylinders, the modified Euclidean metric is  $(1 + \epsilon, \epsilon)$ -bilipschitz to the original Euclidean metric on the traintrack on each branch.*

Now we apply to this modification to our setting. Let  $E'_N$  be the continuous Euclidean metric on  $\mathcal{T}_N$  obtained by making supporting cylinders concentric as above. Similarly let  $E'_t$  be the continuous Euclidean metric on  $\mathcal{T}_t$ . Then, by Proposition 4.5, for every  $\epsilon > 0$ , if  $T$  is sufficiently straight, then  $E'_N$  and  $E'_t$  are piecewise smoothly  $(1 + \epsilon, \epsilon)$ -bilipschitz to  $E_N$  and  $E_t$ , respectively. Therefore, for every  $\epsilon > 0$ , if  $T$  is sufficiently straight, then  $(\mathcal{T}_N, E'_N)$  and  $(\mathcal{T}_t, E'_t)$  are also an  $(\epsilon, K)$ -nearly circular traintrack (for all  $t > 0$ ). Then, since  $E'_N$  and  $E'_t$  are continuous also in the intersection of vertical edges of adjacent branches, by Lemma 4.2 and Lemma 4.4 we have,

**Proposition 4.6.** *There is  $H > 0$  such that, for every  $\epsilon > 0$ , if  $T$  is sufficiently straight, then for sufficiently large  $t > 0$ , the map  $\phi: (\mathcal{T}_N, E'_N) \rightarrow (\mathcal{T}_t, E'_t)$  given by 4.2.1, is a piecewise-differentiable homeomorphism on  $|\mathcal{T}_N|$  and  $(1 + \epsilon, H)$ -bilipschitz on each branch; moreover  $\phi$  is  $\epsilon$ -close to  $\xi: (\mathcal{T}_N, E_N) \rightarrow (\mathcal{T}_t, E_t)$  piecewise smoothly on each branch.*

Clearly, in this proposition, we may in addition assume that  $\phi$  is  $(1 + \epsilon)$ -quasiconformal.

**4.3. Step 3: Gluing the maps between traintracks and their complements.** Notice that  $\partial \mathcal{T}_N$  enjoys different metrics induced from the Euclidean metric  $E'_N$  from  $\mathcal{T}_N$  and from the hyperbolic metric in the complement  $C_N \setminus \mathcal{T}_N$ . Similarly  $\partial \mathcal{T}_t$  enjoys different metrics from the Euclidean metric  $E'_t$  on  $\mathcal{T}_t$  and from the hyperbolic metric in  $C_t \setminus \mathcal{T}_t$ .

**Proposition 4.7.** *For every  $\epsilon > 0$ , if  $T$  is sufficiently straight, then  $\partial\phi: \partial\mathcal{T}_N \rightarrow \partial\mathbb{T}_t$  is  $(1 + \epsilon, \epsilon)$ -bilipschitz with respect to the hyperbolic metric in the complement of the traintracks.*

*Proof.* By Proposition 4.6,  $\xi$  and  $\phi$  are piecewise smoothly  $\epsilon$ -close. Thus it suffices to show  $\partial\xi: \partial\mathcal{T}_N \rightarrow \partial\mathbb{T}_t$  is  $(1 + \epsilon, \epsilon)$ -bilipschitz map.

We first show that  $\partial\xi: \partial\mathcal{T}_N \rightarrow \partial\mathbb{T}_t$  is  $(1 + \epsilon)$ -bilipschitz with respect to the complementary hyperbolic metrics. Let  $\mathcal{R}$  be a branch of the universal cover  $\tilde{\mathcal{T}}_N$  supported on a round cylinder  $\mathcal{A}$ . Let  $\zeta_N: \mathcal{A} \rightarrow c$  be the projection into the core  $c$  of  $\mathcal{A}$ . Let  $H_N$  denote the Euclidean distance in the horizontal direction on a branch  $\mathcal{R}$ , so that  $\zeta_N$  is isometric with  $H_N$  on  $\mathcal{A}$  and the induced hyperbolic metric on  $c$ . Then, if  $T$  is sufficiently straight, since each horizontal edge  $e$  of  $\mathcal{R}$  is nearly orthogonal to the vertical foliation on  $\mathcal{A}$ ,  $H_N$  on  $C$  is  $(1 + \epsilon, \epsilon)$ -bilipschitz to the metric induced from the Euclidean metric  $E_N$  on  $\mathcal{R}$ .

Let  $\beta_N: \mathbb{H}^2 \rightarrow \mathbb{H}^3$  be the pleated surface for  $C_N$  equivariant under  $\text{hol}(C_N)$ . Recall that  $\kappa_N: C_N \rightarrow \tau$  is the composition of the collapsing maps  $C_N \rightarrow C$  and  $C \rightarrow \tau$ . Let  $\tilde{\kappa}_N: \tilde{C} \rightarrow \mathbb{H}^2$  be the lift to a map between their universal covers. Then the restriction of  $\beta_N \circ \tilde{\kappa}_N: \tilde{C}_N \rightarrow \mathbb{H}^3$  to  $e$  is a smooth  $(1 + \epsilon)$ -bilipschitz curve, and it is smoothly  $\epsilon$ -close to the core of  $\mathcal{A}$  (Proposition 3.15). Let  $\mathcal{Y}$  be the stratum of  $(\tilde{C}_N, \tilde{\mathcal{L}})$  containing  $e$ . Then  $\tilde{\kappa}_N$  takes  $\mathcal{Y}$  to a stratum  $Y$  of  $(\mathbb{H}^2, \tilde{\mathcal{L}})$ . Note that  $\text{dev}(C)$  embeds  $\mathcal{Y}$  into  $\hat{\mathbb{C}}$ . Then  $\beta(Y)$  is contained in a hyperbolic plane in  $\mathbb{H}^3$  so that there the nearest point projection onto the hyperbolic plane inducing an isometry  $\eta: \mathcal{Y} \rightarrow \beta(Y)$  with respect to the Thurston metric on  $\mathcal{Y}$ . In particular  $\eta|_e$  is an isometric embedding. Therefore, since  $\beta_N \circ \tilde{\kappa}_N(e)$  is sufficiently close to the core of  $\mathcal{A}$ , if  $T$  is sufficiently straight, then, for each horizontal edge  $e$  of  $\mathcal{T}_N$ , we see that the identity map from  $e$  with  $H_N$  to  $e$  with the complementary hyperbolic metric is  $(1 + \epsilon)$ -bilipschitz.

Similarly, if  $T$  is sufficiently straight, for each horizontal edge of  $\mathbb{T}_t$ , the horizontal Euclidean distance given by  $E_t$  is  $(1 + \epsilon)$ -bilipschitz to the complementary hyperbolic metric.

Recall that  $\xi_i: \mathcal{R}_i \rightarrow \mathcal{R}_i$  induces a linear map  $\xi_{i*}$  in their horizontal coordinates, and, for every  $\epsilon > 0$ , if  $T$  is sufficiently straight, all  $\xi_{i*}$  is  $(1 + \epsilon, \epsilon)$ -bilipschitz with the horizontal distances given by  $E_N$  and  $E_t$ . Since we have seen the horizontal Euclidean distances are bilipschitz to the complementary hyperbolic metrics, hence, if  $T$  is sufficiently straight, for each  $\xi_i: \mathcal{R}_i \rightarrow \mathcal{R}_i$  and each horizontal edge  $e$  of  $\mathcal{R}_i$ , we see that  $\xi_i|_e$  is  $(1 + \epsilon)$ -bilipschitz with the complementary hyperbolic metrics.

Next we show that  $\partial\xi: \partial\mathcal{T}_N \rightarrow \partial\mathbb{T}_t$  is an  $\epsilon$ -rough isometry on each horizontal edge (of each branch). The collapsing maps  $\kappa_N: C_N \rightarrow \tau$  and  $\kappa_t: C_t \rightarrow \tau$  respectively take  $\mathcal{T}_N$  and  $\mathbb{T}_t$  to a single nearly straight traintrack  $T$  on  $\tau$  up to small homotopies of vertical edges. Suppose that  $e_N$  and  $e_t$  are corresponding horizontal edges of  $\mathcal{T}_N$  and  $\mathbb{T}_t$  that descends to a common edge  $e$  of  $T$ . Then  $\kappa_N: C_N \rightarrow \tau$

and  $\kappa_t: C_t \rightarrow \tau$  takes  $e_N$  and  $e_t$  to  $e$  up to  $\epsilon$ -small isotopy (Theorem 3.14). Recall that  $\partial\mathcal{T}_N \cong \partial\mathbb{T}_t$  as projective structures. Then with respect to this identification, if  $T$  is sufficiently straight, then  $e_N$  and  $e_t$  intersect in the curve  $e_N \cap e_t =: h$  and the endpoints of  $h$  are  $\epsilon$  close to the endpoints of  $e_N$  and  $e_t$ .

Let  $\mathcal{A}_N$  and  $\mathbf{A}_t$  be the round cylinders supporting  $e_N$  and  $e_t$ , so that  $e_N \subset \mathcal{A}_N$  and  $e_t \subset \mathbf{A}_t$ . Thus place the supporting cylinders  $\mathcal{A}_N$  and  $\mathbf{A}_t$  on  $\hat{\mathbb{C}}$  so that it realizes the intersection of  $e_N$  and  $e_t$  in  $h$ . In addition, since  $\mathcal{T}_N$  and  $\mathbb{T}_t$  are  $(\epsilon, K)$ -nearly circular,  $e_N$  is contained in a circular rectangle supported on  $\mathcal{A}_N$  of height less than  $\epsilon$ , and  $e_t$  is contained in a circular rectangle of height less than  $\epsilon$  that is supported on  $\mathbf{A}_t$ . Therefore, if  $T$  is sufficiently straight, the cores of  $\mathcal{A}_N$  and  $\mathbf{A}_t$  are  $\epsilon$ -close in  $\mathbb{H}^3$ .

Since  $h$  is  $\epsilon$ -close to  $e_N$  and  $e_t$ , it suffices to show  $\partial\xi$  is  $\epsilon$ -rough isometry on  $h$ . Let  $c_t \subset \mathbb{H}^3$  be the core of  $\mathbf{A}_t$ , and let  $\zeta_t: \mathbf{A}_t \rightarrow c_t$  be the projection along geodesics in  $\mathbb{H}^3$ . Similarly let  $c_N \subset \mathbb{H}^3$  be the core of  $\mathcal{A}_N$ , and let  $\zeta_N: \mathcal{A}_N \rightarrow c_N$  be the projection along geodesics. Since  $c_N$  and  $c_t$  are  $\epsilon$ -close in  $\mathbb{H}^3$ , if  $T$  is sufficiently straight, then for every  $p \in h$ ,  $\zeta_t(p)$  and  $\zeta_N(p)$  are  $\epsilon$ -close in  $\mathbb{H}^3$ . In addition the linear map  $\xi_*: c_t \rightarrow c_N$  is  $\epsilon$ -close to the identity. Therefore, since  $\xi_*$  is an  $\epsilon$ -rough isometry, if  $T$  is sufficiently straight,  $\xi_*$  takes  $\zeta_t(p)$  to a point  $\epsilon$ -close to  $\zeta_N(p)$  for each  $p \in h$ . Since the hyperbolic metrics are  $\epsilon$ -rough isometric to the horizontal metrics,  $\xi$  is  $\epsilon$ -rough isometric on  $h$  with the complementary hyperbolic metrics.  $\square$

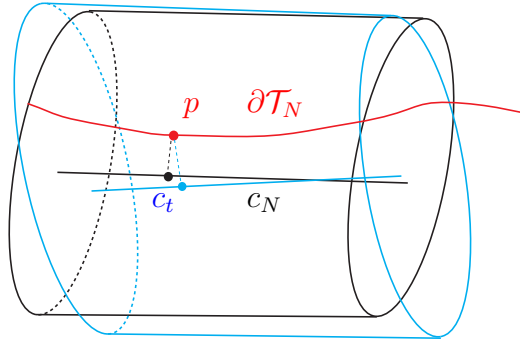


FIGURE 5.

4.3.1. *Matching  $\phi$  and  $\psi$  along the boundary of the traintrack.* Although  $\phi: \mathcal{T}_N \rightarrow \mathbb{T}_t$  and  $\psi: C_N \setminus \mathcal{T} \rightarrow C_t \setminus \mathbb{T}$  do not coincide along the boundary of  $\mathcal{T}_N$ , however by Proposition 4.7, for every  $\epsilon > 0$ , if  $T$  is sufficiently straight, then  $\psi \circ \phi^{-1}$ , defined on  $\partial\mathbb{T}_t$ , is  $(1 + \epsilon, \epsilon)$ -bilipschitz map (with either metrics). Thus, for sufficiently

large  $t > 0$ , we modify  $\phi$  so that  $\phi$  and  $\psi$  agree along  $\partial\mathcal{T}$  by post-composing  $\phi$  with  $(1 + \epsilon, \epsilon)$ -bilipschitz map.

Let  $\mathbf{N}(= \mathbf{N}_t)$  be the *unit collar* of  $\partial\mathbb{T}_t$  in  $\mathbb{T}_t$ . That is, the union of the 1-neighborhoods of the horizontal edges in the branches of  $\mathbb{T}_t$  (Figure 6) with respect to the modified euclidean metric  $E'_t$  included by concentric cylinders on  $\hat{\mathbb{C}}$  (§4.2.3).

Recall that if  $t > 0$  is sufficiently large, then all branches of  $\mathbb{T}_t$  has height more than 2. Then  $\mathbf{N}$  is a union of finitely many Euclidean cylinders and rectangles of height 1 that are disjoint except at vertices on the switch points on  $\mathbb{T}_t$ . Thus set  $\mathbf{N} = \cup_j \mathbf{X}_j$ , where  $\mathbf{X}_j$  are the Euclidean cylinders and rectangles.

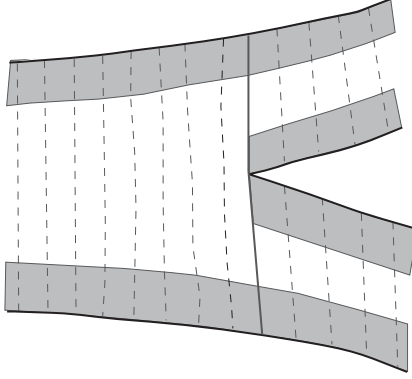


FIGURE 6.

If  $\mathbf{X}_j$  is a rectangle, let  $a_j$  be its horizontal edge contained in the interior of  $\mathbb{T}_t$  and  $b_j$  be its horizontal edge in the boundary of  $\mathbb{T}_t$ . Similarly, if  $\mathbf{X}_j$  is a cylinder, let  $a_j$  be its boundary circle contained in the interior of  $\mathbb{T}_t$  and  $b_j$  be the other boundary circle contained in the boundary of  $\mathbb{T}_t$ . Note that  $\psi \circ \phi^{-1}: \mathbb{T}_t \rightarrow \mathbb{T}_t$  preserves each  $b_j$ .

Since  $\psi \circ \phi^{-1}$  on  $\partial\mathbb{T}_t$  is a  $(1 + \epsilon, \epsilon)$ -bilipschitz map, we can construct a desired bilipschitz map supported on  $\mathbf{N}$ : The vertical foliation of  $\mathbb{T}_t$  induces a vertical foliation on each  $\mathbf{X}_j$ , each leaf of which connects a point on  $a_j$  and a point on  $b_j$ . Then there is a unique homeomorphism  $\eta_j: \mathbf{X}_j \rightarrow \mathbf{X}_j$  such that

- $\eta_j$  is the identity map on  $a_j$ ,
- $\eta_j$  is  $\psi \circ \phi^{-1}$  on  $b_j$ , and
- $\eta_j$  is linear on each vertical leaf.

Since,  $\psi \circ \phi^{-1}$  on  $b_j$  is a piecewise differentiable homeomorphism, so is  $\eta_j$ . Then, for every  $\epsilon > 0$ , if  $T$  is sufficiently straight, then for sufficiently large  $t > 0$ ,  $\eta_j: \mathbf{X}_j \rightarrow \mathbf{X}_j$  are  $(1 + \epsilon, \epsilon)$ -bilipschitz. Let  $\eta: \mathcal{T}_N \rightarrow \mathbb{T}_t$  be the homeomorphism



such that  $\eta = \eta_j$  on each  $X_j$  and the identity map on  $\mathbb{T}_t \setminus \mathbb{N}$ . Then  $\eta$  is  $(1 + \epsilon, \epsilon)$ -bilipschitz and  $\eta \circ \phi = \psi$  on  $\partial\mathcal{T}_N$ .

Since  $\eta \circ \phi: \mathcal{T}_N \rightarrow \mathbb{T}_t$  and  $\psi: C_N \setminus \mathcal{T}_N \rightarrow C_t \setminus \mathbb{T}_t$  are both  $(1 + \epsilon, \epsilon)$ -bilipschitz, we obtain a desired  $(1 + \epsilon, \epsilon)$ -bilipschitz map  $C_N \rightarrow C_t$  for sufficiently large  $t > 0$ .

## REFERENCES

- [Ahl66] Lars V. Ahlfors, *Lectures on quasiconformal mappings*, Manuscript prepared with the assistance of Clifford J. Earle, Jr. Van Nostrand Mathematical Studies, No. 10, D. Van Nostrand Co., Inc., Toronto, Ont.-New York-London, 1966.
- [Baba] Shinpei Baba,  *$2\pi$ -graftings and complex projective structures I*, preprint, *arXiv:1011.5051*.
- [Babb] ———,  *$2\pi$ -graftings and complex projective structures II*, preprint, *arXiv:1307.2310*.
- [CB88] Andrew J. Casson and Steven A. Bleiler, *Automorphisms of surfaces after Nielsen and Thurston*, London Mathematical Society Student Texts, vol. 9, Cambridge University Press, Cambridge, 1988. MR MR964685 (89k:57025)
- [CEG87] R. D. Canary, D. B. A. Epstein, and P. Green, *Notes on notes of Thurston*, Analytical and geometric aspects of hyperbolic space (Coventry/Durham, 1984), London Math. Soc. Lecture Note Ser., vol. 111, Cambridge Univ. Press, Cambridge, 1987, pp. 3–92. MR 903850 (89e:57008)
- [Dum09] David Dumas, *Complex projective structures*, Handbook of Teichmüller theory. Vol. II, IRMA Lect. Math. Theor. Phys., vol. 13, Eur. Math. Soc., Zürich, 2009, pp. 455–508. MR MR2497780
- [EM87] D. B. A. Epstein and A. Marden, *Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces*, Analytical and geometric aspects of hyperbolic space (Coventry/Durham, 1984), London Math. Soc. Lecture Note Ser., vol. 111, Cambridge Univ. Press, Cambridge, 1987, pp. 113–253. MR 903852 (89c:52014)
- [GKM00] Daniel Gallo, Michael Kapovich, and Albert Marden, *The monodromy groups of Schwarzian equations on closed Riemann surfaces*, Ann. of Math. (2) **151** (2000), no. 2, 625–704. MR MR1765706 (2002j:57029)
- [Gol80] William M. Goldman, *Discontinuous groups and the Euler class*, Ph.D. thesis, 1980.
- [Gol87] ———, *Projective structures with Fuchsian holonomy*, J. Differential Geom. **25** (1987), no. 3, 297–326. MR MR882826 (88i:57006)
- [Gol04] ———, *The complex-symplectic geometry of  $SL(2, \mathbb{C})$ -characters over surfaces*, Algebraic groups and arithmetic, Tata Inst. Fund. Res., Mumbai, 2004, pp. 375–407. MR 2094117 (2005i:53110)
- [Gup] Subhojoy Gupta, *Asymptoticity of grafting and teichmüller rays I*, Geom. Topol. (to appear).
- [Gup14] ———, *Asymptoticity of grafting and Teichmüller rays II*, Geom. Dedicata. (to appear) (2014).
- [Hej75] Dennis A. Hejhal, *Monodromy groups and linearly polymorphic functions*, Acta Math. **135** (1975), no. 1, 1–55. MR MR0463429 (57 #3380)
- [Hub81] John H. Hubbard, *The monodromy of projective structures*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), Ann. of Math. Stud., vol. 97, Princeton Univ. Press, Princeton, N.J., 1981, pp. 257–275. MR MR624819 (82k:14009)

- [Hub06] John Hamal Hubbard, *Teichmüller theory and applications to geometry, topology*, Matrix Editions, Ithaca, NY, 2006.
- [Kap95] Michael Kapovich, *On monodromy of complex projective structures*, Invent. Math. **119** (1995), no. 2, 243–265. MR MR1312500 (95k:57017)
- [Kap01] ———, *Hyperbolic manifolds and discrete groups*, Progress in Mathematics, vol. 183, Birkhäuser Boston Inc., Boston, MA, 2001. MR MR1792613 (2002m:57018)
- [KP94] Ravi S. Kulkarni and Ulrich Pinkall, *A canonical metric for Möbius structures and its applications*, Math. Z. **216** (1994), no. 1, 89–129. MR MR1273468 (95b:53017)
- [KT92] Yoshinobu Kamishima and Ser P. Tan, *Deformation spaces on geometric structures*, Aspects of low-dimensional manifolds, Adv. Stud. Pure Math., vol. 20, Kinokuniya, Tokyo, 1992, pp. 263–299. MR MR1208313 (94k:57023)
- [Mas82] Howard Masur, *Interval exchange transformations and measured foliations*, Ann. of Math. (2) **115** (1982), no. 1, 169–200. MR 644018 (83e:28012)
- [PH92] R. C. Penner and J. L. Harer, *Combinatorics of train tracks*, Annals of Mathematics Studies, vol. 125, Princeton University Press, Princeton, NJ, 1992. MR MR1144770 (94b:57018)
- [Tan99] Harumi Tanigawa, *Divergence of projective structures and lengths of measured laminations*, Duke Math. J. **98** (1999), no. 2, 209–215. MR 1695198 (2000c:32047)

CALIFORNIA INSTITUTE OF TECHNOLOGY, MATHEMATICS 253-37, PASADENA CA 91125  
USA

*E-mail address:* shinpei@caltech.edu and subhojoy@caltech.edu